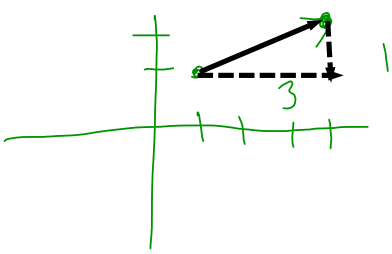
6.6 Vectors (directed line segments)

Vector: magnitude and direction

slope of vector

distance from the initial point to the terminal point



$$d = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

or Pythagorean Th.

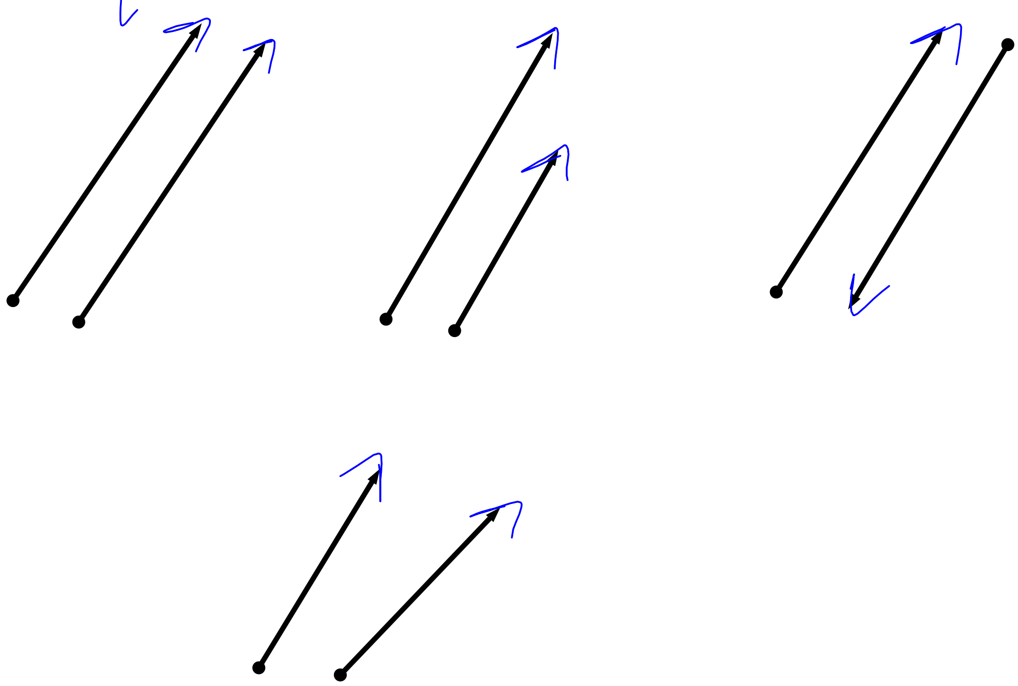
Scalar: just a number, only magnitude
no direction

Notation: $\vec{v}$: vector v

$\|\vec{v}\|$: magnitude of $\vec{v}$

Equal Vectors: Same magnitude and same direction

equal



$\vec{u}$: Initial Point: $(-3,-3)$

Terminal Point: $(0,3)$

$\vec{v}$: Initial Point: $(0,0)$

Terminal Point: $(3,6)$

Show that $\vec{u}=\vec{v}$

$\vec{u}$: $d=\sqrt{(-3-0)^2+(-3-3)^2}$

$\sqrt{(-3)^2+(-6)^2}=\sqrt{9+36}=\sqrt{45}$

$\sqrt{45}=3\sqrt{5}$

$\vec{v}$: $\Delta x=3$ $\Delta y=6$

$||\vec{v}||=\sqrt{9+36}=\sqrt{45}=3\sqrt{5}$

direction: slope $=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{\Delta y}{\Delta x}$

$\vec{u}$ slope $=\dfrac{-3-3}{-3-0}=\dfrac{-6}{-3}=2$

$\vec{v}$ slope $=\dfrac{6-0}{3-0}=\dfrac{6}{3}=2$

Magnitude and direction are the same, so $\vec{u}=\vec{v}$

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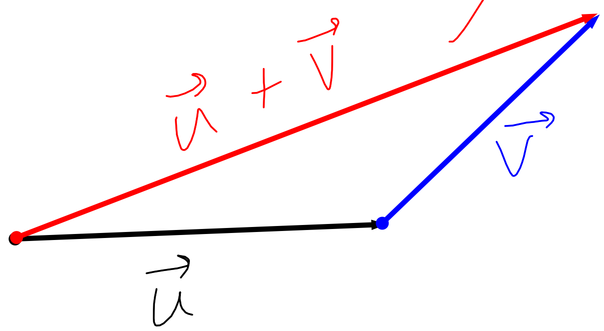
Scalar multiplication: multiply the magnitude by a number

positive scalar: change magnitude, but not direction

negative scalar: changes magnitude, flips the direction

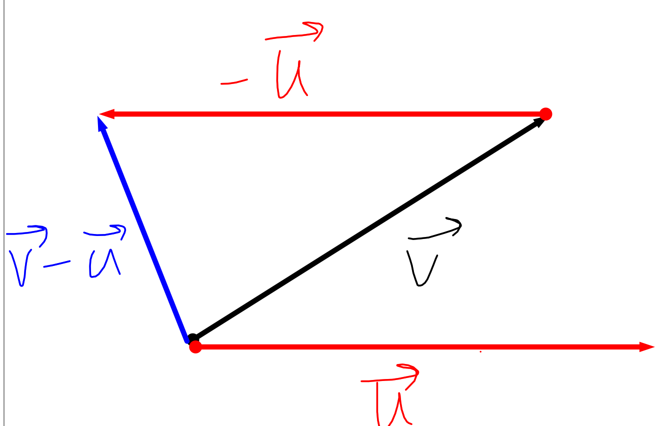
Vector addition

resultant vector



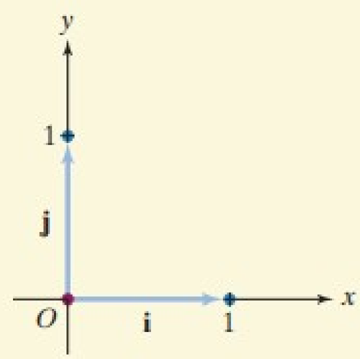
vector addition: intial point of one vector is the same as the terminal point of another.

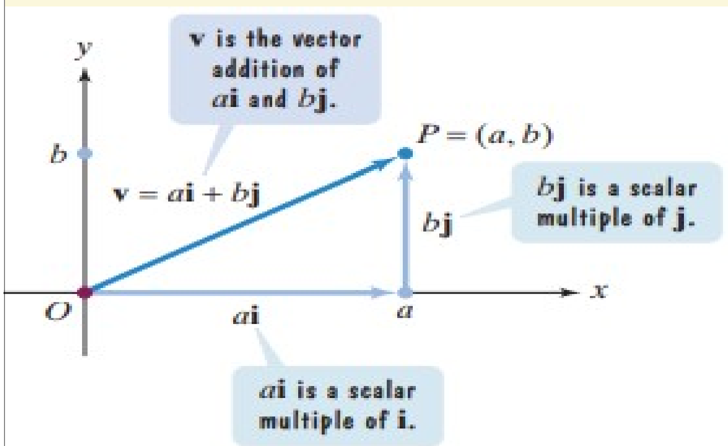
$$\vec{v}-\vec{u}=\vec{v}+(-\vec{u})$$



The i and j Unit Vectors

Vector $\mathbf{i}$ is the unit vector whose direction is along the positive x-axis. Vector $\mathbf{j}$ is the unit vector whose direction is along the positive y-axis.





$$v = ai + bj$$

Initial point:
(x_1, y_1)

$a = \Delta x$ $b = \Delta y$
$(x_2 - x_1)$ $(y_2 - y_1)$

Terminal point:
(x_2, y_2)

$v = \langle a, b \rangle$ this notation is always the vector with the initial point at the origin.

$$\|v\| = \sqrt{a^2 + b^2}$$ magnitude

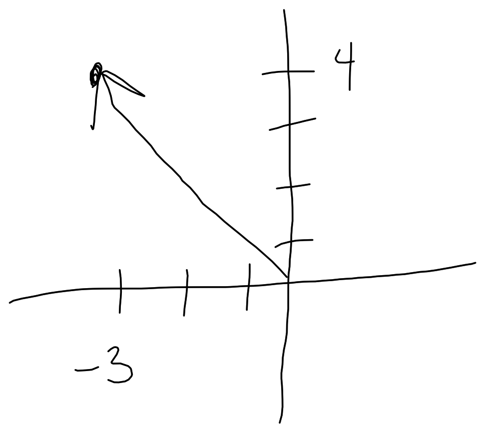
EXAMPLE 2 Representing a Vector in Rectangular Coordinates and Finding Its Magnitude

Sketch the vector $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$ and find its magnitude.

$$\|v\| = \sqrt{(-3)^2 + (4)^2}$$

$$\sqrt{25} = 5$$

$$\|v\| = 5$$



EXAMPLE 3 Representing a Vector in Rectangular Coordinates

Let **v** be the vector from initial point $P_1 = (3, -1)$ to terminal point $P_2 = (-2, 5)$. Write **v** in terms of **i** and **j**.

$x_1\ y_1$ $x_2\ y_2$

$a = \Delta x = (x_2 - x_1) = -2 - 3 = -5$

$b = \Delta y = (y_2 - y_1) = 5 - (-1) = 6$

$v = -5i + 6j$

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Check Point 3 Let **v** be the vector from initial point $P_1 = (-1, 3)$ to terminal point $P_2 = (2, 7)$. Write **v** in terms of **i** and **j**.

$a = 2-(-1) = 3$

$b = 7-3 = 4$

$v = 3i + 4j$

Adding and Subtracting Vectors in Terms of i and j

combine like terms

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$$

$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}.$$

Scalar Multiplication with a Vector in Terms of i and j

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and k is a real number, then the scalar multiplication of the vector $\mathbf{v}$ and the scalar k is

$$k\mathbf{v} = (ka)\mathbf{i} + (kb)\mathbf{j}.$$

Distribute

EXAMPLE 4 Adding and Subtracting Vectors

If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 9\mathbf{j}$, find each of the following vectors:

a. $\mathbf{v} + \mathbf{w}$ **b.** $\mathbf{v} - \mathbf{w}$.

a) $v + w = (5i + 4j) + (6i - 9j)$

$v + w = 11i - 5j$

b) $v - w = (5i + 4j) - (6i - 9j)$

$5i + 4j - 6i + 9j$

$-i + 13j$

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EXAMPLE 6 Vector Operations

If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 9\mathbf{j}$, find $4\mathbf{v} - 2\mathbf{w}$.

$4v = 4(5i + 4j) = 20i + 16j$

$-2w = -2(6i - 9j) = -12i + 18j$

$8i + 34j$

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Check Point 6 If $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 5\mathbf{j}$, find $6\mathbf{v} - 3\mathbf{w}$.

$30i + 33j$

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The Zero Vector

The vector whose magnitude is 0 is called the **zero vector, 0**. The zero vector is assigned no direction. It can be expressed in terms of **i** and **j** using

$$\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}.$$

Properties of Vector Addition and Scalar Multiplication

If **u**, **v**, and **w** are vectors, and c and d are scalars, then the following properties are true.

Vector Addition Properties

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative property
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ — Additive identity
4. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$ — Additive inverse

Scalar Multiplication Properties

1. $(cd)\mathbf{u} = c(d\mathbf{u})$ — Associative property
2. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive property
3. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive property
4. $1\mathbf{u} = \mathbf{u}$ — Multiplicative identity
5. $0\mathbf{u} = \mathbf{0}$ — Multiplication property of zero
6. $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$ — Magnitude property

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A unit vector is defined to be a vector whose magnitude is one.

Finding the Unit Vector that Has the Same Direction as a Given Nonzero Vector v

For any nonzero vector **v**, the vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is the unit vector that has the same direction as **v**. To find this vector, divide **v** by its magnitude.

EXAMPLE 7 Finding a Unit Vector

Find the unit vector in the same direction as $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$. Then verify that the vector has magnitude 1.

$$\|v\| = \sqrt{5^2 + (-12)^2}$$

$$\sqrt{25 + 144} = \sqrt{169} = 13$$

$$\|v\| = 13$$

$$\text{unit vector} = \frac{v}{\|v\|} = \frac{5i - 12j}{13}$$

$$a = \frac{5}{13} \qquad \text{unit vector } \frac{5}{13}i - \frac{12}{13}j$$

$$b = -12/13$$

$$\|\text{unit vector}\| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(-\frac{12}{13}\right)^2}$$

$$\sqrt{\frac{25}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = 1$$

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Check Point 7 Find the unit vector in the same direction as $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Then verify that the vector has magnitude 1.

$$\text{unit vector} = u = \frac{v}{\|v\|}$$

$$\|v\| = \sqrt{a^2 + b^2} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

$$u = \frac{4i - 3j}{5} = \frac{4}{5}i - \frac{3}{5}j$$

$$\|u\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{1} = 1$$

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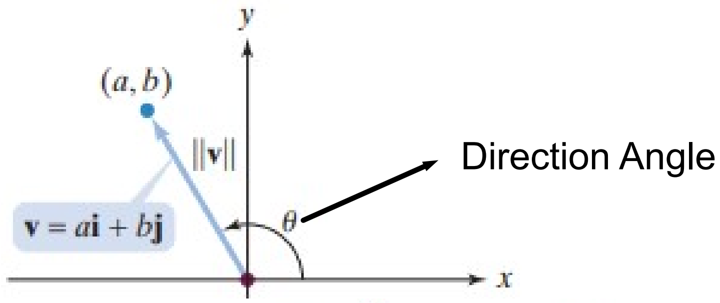


$$\cos\theta = \frac{a}{\|\mathbf{v}\|} \quad \text{and} \quad \sin\theta = \frac{b}{\|\mathbf{v}\|}$$

$$a = \|\mathbf{v}\|\cos\theta \qquad b = \|\mathbf{v}\|\sin\theta.$$

Thus,

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j}.$$

Writing a Vector in Terms of Its Magnitude and Direction

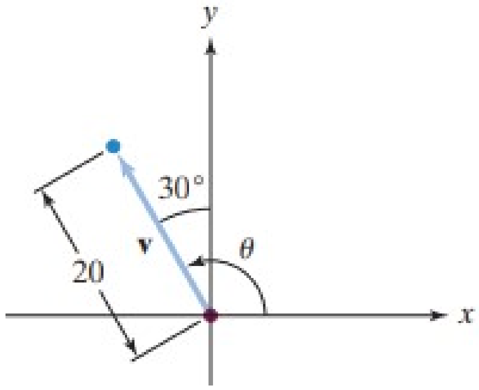
Let $\mathbf{v}$ be a nonzero vector. If θ is the direction angle measured from the positive x-axis to $\mathbf{v}$, then the vector can be expressed in terms of its magnitude and direction angle as

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j}.$$

A vector that represents the direction and speed of an object in motion is called a **velocity vector**

Example:

The wind is blowing at 20 miles per hour in the direction N30°W. Express its velocity as a vector **v** in terms of **i** and **j**.



$\|v\| = 20$ mph

$\theta = 90 + 30 = 120$

$a = \|v\| \cos\theta$

$a = 20 \cos 120 = 20\left(-\frac{1}{2}\right)$

$a = -10$

$b = \|v\| \sin\theta = 20 \sin 120$

$20\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$

$v = -10i + 10\sqrt{3}\, j$

Example:

$u = 3i - 6j \qquad v = -2i + j$

Find $\|u - v\|^2 + \|u + v\|^2$

$u - v = (3i - 6j) - (-2i + j)$

$3i - 6j + 2i - j = 5i - 7j$

$u + v = (3i - 6j) + (-2i + j) = i - 5j$

$\|u - v\|^2 = \left(\sqrt{5^2 + (-7)^2}\right)^2 = 74$

$\|u + v\|^2 = \left(\sqrt{1^2 + (-5)^2}\right)^2 = 26$

$74 + 26 = 100$

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Assignment pgs. 750-751:

#1, 5, 9, 13, 17, 23, 27, 31, 37, 41, 45, 47, 49

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